6.1. Conjugate points. Let $c\colon [a,b] \to M$ be a geodesic such that for all $t \in (a,b]$ the point $c(t)$ is not conjugate to $c(a)$ along c. Show that for all $s,t \in [a,b]$ with $s < t$, we have that $c(t)$ is not conjugate to $c(s)$ along c.

6.2. Trace of a symmetric bilinear form. Let $(V, \langle\cdot,\cdot\rangle)$ be a m-dimensional Euclidean space and let $r\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form. Furthermore, let $S^{m-1} = \{v \in V : |v| = 1\}$ be the unit sphere. Prove that

$$\int_{S^{m-1}} r(v,v)\ d\mathrm{vol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\mathrm{tr}(r) = \omega_m \mathrm{tr}(r),$$

where $d\mathrm{vol}^{S^{m-1}}$ denotes the induced volume on S^{m-1} and ω_m is the volume of the m-dimensional unit ball.

6.3. Small balls and scalar curvature. Let p be a point in the m-dimensional Riemannian manifold (M,g). To goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\mathrm{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{1}{6(m+2)}\mathrm{scal}(p) r^2 + \mathcal{O}(r^3)\right).$$

1. Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $v, e_2, \dots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \dots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by
$$J(v,t) := \sqrt{\det\left(\langle T_{tv} e_i, T_{tv} e_j\rangle\right)} = t^{-(m-1)}\sqrt{\det\left(\langle Y_i, Y_j\rangle\right)},$$
where $T_{tv} := (d\exp_p)_{tv}$.
2. Let $E_2, \dots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is
$$Y_i(t) = tE_i - \sum_{k=2}^{m}\left(\tfrac{t^3}{6} R(e_i, v, e_k, v) + \mathcal{O}(t^4)\right) E_k.$$
3. Conclude that $J(v,t) = 1 - \frac{t^2}{6}\mathrm{ric}(v,v) + \mathcal{O}(t^4)$.
   Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon\mathrm{tr}(A) + \mathcal{O}(\epsilon^2)$.
4. Prove the above formula for $\mathrm{vol}(B_r(p))$.